# A new class of scale free solutions to linear ordinary differential equations and the universality of the Golden Mean $\frac{\sqrt{5}-1}{2}$

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**Abstract**

A new class of finitely differentiable scale free solutions to the simplest class of ordinary differential equations is presented. Consequently, the real number set gets replaced by an extended physical set, each element of which is endowed with an equivalence class of infinitesimally separated neighbours in the form of random fluctuations. We show how a sense of time and evolution is intrinsically defined by the infinite continued fraction of the golden mean irrational number $\frac{\sqrt{5}-1}{2}$, which plays a key role in this extended SL(2,R) formalism of Calculus. Time may thereby undergo random inversions generating well defined random scales, thus allowing a dynamical system to evolve self similarly over the set of multiple scales. The late time stochastic fluctuations of a dynamical system enjoys the generic $1/f$ spectrum. A universal form of the related probability density is also derived. We prove that the golden mean number is intrinsically random, letting all measurements in the physical universe fundamentally uncertain. The present analysis offers an explanation of the universal occurrence of the golden mean in diverse natural and biological processes.

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# 1 Introduction

In Calculus, a variable changes by ordinary shift operations (translations). Every point in the real axis is conceived as a structureless point. In a dynamical problem, the time evolution of a system occurs over a finite number of characteristic scales as inherited from the underlying differential equation. The generation of fractal- like self similar structures over multiple (dynamically generated) scales, either in the phase space or purely in the time sector, needs explicit nonlinearity at the level of the governing equation. In the following, we present a new class of scale free stochastic solutions of the simplest ordinary differential equation. A representative solution of the new class, which is exact and finitely differentiable, defines an extension of the real number set, endowing every number with a nontrivial neighbourhood of fluctuations. These fluctuations experience an intrinsic universal evolution, which, in turn, generates a generic irreversible sense of time defined by the infinite continued fraction representation of the 'golden mean' irrational number. Consequently, time may undergo random inversions, allowing a dynamical system to explore stochastically, new scales, usually unavailable in the ordinary dynamics. The corresponding evolution of the system would, therefore, enjoy a generic late time stochastic fluctuations with $1/f$ spectrum [1]. We also obtain an exact form of the universal probability density, recently found [2] to occur naturally in a wide class of scale free nonlinear processes in Nature. These exact results provide a natural explanation of the universal occurrence of the golden mean in diverse natural and biological processes.

The relevance of the present work may also be seen in the light of the recent works [3,4] uncovering new relationships between time and the number theory. In ref [3], the evidence of $1/f$ noise and scale free self similarity in the prime number distribution is pointed out. In ref [4], Planat reported the observation of discrete time jumps in beat frequencies in the context of a superheterodyne receiver. These random jumps are shown to generate $1/f$ noise in the oscillator frequencies. Further, the origin of $1/f$ spectrum is related to the arithmetical summatory functions such as the Mobius and the Mangoldt functions which arise in connection with the Reimann zeta function. Sometime back, Robinson, on the other-hand, developed [5] a theory of infinitesimals on the basis of mathematical logic, thereby giving rise to a valid model of a nonstandard extension of the standard framework of analysis. As it turns out, the present extension of Calculus is a realization of the nonstandard analysis, in which infinitesimally small numbers are shown to have dynamical properties. We note also that our exact results seem to provide a correct mathematical framework for the theories of fractal space-time being developed by several authors [6,7,8] . A significant part of El Naschie's work [8], in particular, explores the

significance of the golden mean in Cantorian fractal spacetime and other related branches of Physics and Mathematics. A detailed comparison of the present results with those of Refs.[6-8] would require separate investigations. Incidentally, the possibility of fractal solutions to linear differential equations seems to have been pointed out first by Nottale [6].[1]

To indicate how the exact results, reported here, are obtained, starting from a heuristic definition of time inversion, we present the exact results in Sec.4. In the preceding two sections we indicate how the exact class of solutions is derived in Sec.3, using an approximate analysis, based on a local definition of time inversion, explained in Sec.2. Applications of the results are discussed in Sec.5. We close the presentation with some further remarks pointing out the future scope of this extended formalism of Calculus.

# 2 Time inversion

Let us consider the simplest linear dynamical system given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1+\kappa t)x \tag{1}$$

where $t$ denotes the dimensionless time ( we scale $t$ suitably to adjust the dominant scale of evolution to $t \sim 1$) and $\kappa$ is a small, slowly varying (almost constant), parameter. In the ordinary Calculus, this equation has the 'standard' solution $x_s \propto \exp(t + \frac{1}{2}\kappa t^2)$, with no (self-similar) fluctuations, unless the equation is explicitly nonlinear, for instance, through $\kappa = \kappa(t, x, \dot{x}, \ddot{x}, \ldots)$. Our intention is to show that eq (1) can accommodate a new class of nonlinear stochastic solutions, even for a constant $\kappa$, under a simple, but general assumption that *time may change by inversions as well.*

Let us recall that ordinarily a change in time, in the vicinity of a given instant $t_0$ is indicated by a pure translation $t = t_0 + \bar{t} \equiv t_0 + (t - t_0)$. In fact, the last equality is an identity (valid for all $t$). By an inversion, on the other hand, we mean the following. Let $t_\pm$ denote times $t \lessapprox 1$ and $t \gtrapprox 1$ respectively (from now on $t$ denotes the rescaled variable $t \to \frac{t}{t_0}$). Then close to $t = 1$, the inversion $t_- = 1/(1 + (t_+ - 1))$ leads to the *constraint* $1 - t_- = t_+ - 1$. The parametric representation of inversely related times is obviously given by $t_- = 1 - \bar{t}$ and $t_+ = 1 + \bar{t}$, $0 < \bar{t} << 1$ (so that the constraint reduces to an *identity*, valid close to $t = 1$). With this reinterpretation, time inversion in the vicinity of $t = 1$ assumes a form analogous to a pure translation. If translation is considered to be the most natural mode of time increment, then there is no compelling reason of ignoring inversion as yet another *natural* mode of doing

[1] Prof. El Naschie has pointed out that it is in-fact G. N. Ord who first conceived the possibility of fractal solutions to linear differential equations.

this. Consequently, it seems reasonable to assume that time may change from $t_-$ to $t_+$ not only by ordinary translation over the period $t_+ - t_- = 2\bar{t}$, but also instantaneously by an inversion. Let us clarify the physical content of the definition further through several remarks.

1. Note that the above definition gives a *new nontrivial* solution $t_- = 1/t_+$ to the constraint $t_- + t_+ = 2$, in the *vicinity* of $t = 1$ over the linear solution $t_- = 2 - t_+$, ordinarily thought to be the only possible solution. In view of this nonlinear possibility, the change (flow) of time could be visualised as an SL(2,R) group action, when the nontrivial SL(2,R) action is realized only in a neighbourhood of a point, $t = 1$, say. One may thus imagine that time flows, for example, from $t = 0$ to $t = t_-$ by translation, and then may switch over to $t_+$ by inversion $t_+ = 1/t_-$, for another period of linear flow etc. In the next sections, we show how this second and subsequent periods of linear flows are actually realized over scales of the form $t_n = \nu^n t$, $n = 1, 2, \ldots$, where $\nu = \frac{\sqrt{5}-1}{2} = 0.618033939...$, is the golden mean irrational number.[2]

2. The definition of time inversion has an inbuilt uncertainty. The moments $t_\pm$ are not well defined, except for the fact that these be close to $t = 1$, thus elevating $\bar{t}$, and hence time itself, to the status of a random variable. Note that this sort of uncertainty (randomness) is not allowed in the framework of linear Newtonian time. In the present SL(2,R) framework, time may, therefore, undergo small scale random fluctuations in the neighbourhood of every instant ($\bar{t} << 1$). Two instants $t_\pm$ joined by an inversion, can therefore be in a continual process of transport between each other by randomly flipping their signs. The possibility of random flipping also indicates that there is an inherent uncertainty in the actual determination of an instant close to $t = 1$. In case one claims that he/she is in the moment $t_-$, then there is a 50% chance that he/she is actually at the moment $t_+$. From now on, we denote by $t$ this stochastic behaviour of time, when the ordinary Newtonian time is denoted by $\eta$.

3. The ordinary nonrandom variable (time) $\eta$, that we are accustomed to at moderate scales should be retrievable from the stochastic $t$ in the mean $\eta = < t >$ (upto a rescaling). Physically, it means that the small scale fluctuations near every point of $t$ cancel each other in such a way to yield the average *coarse-grained* time sense $\eta$ to our experience. These fluctuations would, however, become important to determine the small scale structure of time and hence of a dynamical system. By inversion, these would also have nontrivial influences on the long time behaviour of the system. Incidentally, we note that the formal definition of inversion is still valid for $\eta$. For a nonzero mean $\bar{t}$, for instance, inversion between $\eta_\pm = 1 \pm < \bar{t} >$ could

[2]The standard notation of the golden mean in the literature is $\phi$. Our symbol $\nu$ in this work is motivated by the fact that the golden mean appears here as a universal scaling factor of time. The symbol $\phi$ is more often used to denote a quantum state function as well. In Sec.4-5, we, however, denote the stochastic time-like character of the golden mean by $\phi$.

then be implemented at a well determined instant. For a zero mean $\bar{t}\,(<\bar{t}>=0)$ , on the other hand, the inversion constraint collapses to $\eta = 1$. Thus allowing for a random, zero mean $\bar{t}$ is equivalent to probing small scale structure of a point, $t = 1$, where by point we mean a 'physical point' which is meaningful only in the context of an accuracy limit (level of resolution), in contrast to an ideal structureless point . As an example, let us consider a situation when the measurement of a duration (an interval) is allowed only upto the first decimal accuracy. In that case, points separated by a distance less than .05 are indistinguishable, and should be treated as an equivalence class. In this sense, the point $t = 1$ is an equivalent class, where the symbol '1' is only a convenient representative of the class. In the ideal case of infinite accuracy, the class '1' has only one element in the ordinary (linear) Calculus. In the presence of inversions, it now follows that even an ideal point ( in the limit of infinite accuracy) has infinite number of nontrivial members separated by infinitesimally small, zero mean random scales. Since $\bar{t}(t)$ is a positive, monotonically increasing stochastic variable, the nontrivial equivalence class of an ideal point has, at the least, the cardinality of continuum. In fact, it can be higher, as is shown in Sec.4. (It will be evident that zero mean is not a necessary restriction.)

4. It turns out that the concepts of inversion and stochastic time are deeply related to the (practical ) limitation on the exact measurability of a duration. Let, for instance, the moments $t_-$, $t_+$ belong to the equivalence class of the 'physical point' $t = 1$. Then for any practical purpose, $t_- = t_+ = 1$. That means, in turn, that any measurement over a finite period of time would fail to make any distinction between $t_-$, $t_+$ and 1. The associated fluctuations between $t_-$, $t_+$ which remain imperceptible over a pretty long time can, however, act as potential seeds for self-similar evolutions over scales $t_n = \nu^n t$, which can grow slowly to interfere with each other, thereby deflecting the standard nonrandom evolution, to a universal pattern of stochastic fluctuations (c.f.,Sec.3).

5. It is worth comparing the present definition of a local time inversion with the usual (global) time reversal (inversion) symmetry of an equation of the form eq(1). The usual time reversal symmetry means that the system $x(t)$ evolves not only forward in time from $t$ to $t+h, h > 0$, but it can also evolve backward; i.e., the state $x(t)$ can be reconstructed from the state $x(t+h)$. The parameter $t$ in eq(1) is thus 'non-directed', giving rise to the problem of time asymmetry. As remarked already, the new class of solutions indicates an inherent irreversibility in the time sense.

# 3 Fractal Solution

In the framework of SL(2,R) stochastic time, eq (1), rewritten as

$$\mathrm{d}x = (1+\kappa t)x\,\mathrm{d}t \tag{2}$$

assumes the status of a stochastic differential equation. We now present a treatment of the above equation when the variable $t$ is assumed to act as the ordinary 'coarse grained' time $\eta$, except in the vicinity of $t = 1$ (the dominant scale of evolution). Thus over the period (0,1), the system evolves along the standard evolutionary path $x_0 \equiv x_s \approx e^{\eta}$ ($\kappa$ being small). The small scale fluctuations in the system variable $x$, as inherited from the fluctuations in $t$ would remain un-noticeable in comparison with this mean evolution. One can, however, probe these fluctuations, provided the mean evolution $x_0$ is removed from the actual (total) system $x$ via the ansatz $x = x_0 x_1$, where $x_1$ is the purely 'fluctuating' component satisfying the reduced equation

$$\mathrm{d}x_1 = \kappa t\, x_1 \mathrm{d}t \tag{3}$$

To prove the self similarity of the small scale fluctuations, and to see how these grow and influence the late time properties of the mean evolution, we probe the neighbourhood of $t = 1$ more and more closely, using scales $t_n$, separating in each step the relevant residual evolution into 'mean' and 'fluctuation' respectively. To see this explicitly, let us assume, for definiteness, that the actual removal of the mean in eq(2) is accomplished at an instant $t_- \lessapprox 1$. We note ,however, that the specification of the exact moment is physically impossible. Thus the moment $t_-$ must have an inherent uncertainty, which we represent by an ansatz $t_- = 1 - r\bar{\eta}$, where, $r$ is a discrete variate with moments $< r^n > \approx r_0^n$, $< r >= r_0$, so that $< t_- >= \eta_-$. For the sake of clarity, one may also assume that $t_+$ and $t_-$ belong to the physical equivalence class of 1(c.f., remark 4), when a point is defined correct to $m$ decimals (so that the accuracy limit is $10^{-m}$). Because of the random inversions $t_+ \leftrightarrow t_-$ in the time coordinate, the residual system pair $x_1(t_-)$ and $x_1(t_+)$ also undergo fluctuations between them. However, these fluctuations would remain unobservable at least over a sufficiently long period of the form $0 < \eta < 1 + a$, provided $r_0$ is sufficiently small and $a \sim 1/(10^m r_0)$. *Now utilising the golden mean partition of unity*: $\nu^2 + \nu = 1$, $\nu > 0$, one can realise a scale changing SL(2,R) transformation

$$t_+ = 1 + r\bar{\eta} \approx \frac{1 + \tilde{r}\nu\,\bar{\eta}_1}{1 - \tilde{r}\bar{\eta}_1} \tag{4}$$

where $\bar{\eta}_1 = r_0\nu\bar{\eta} = \nu(\eta_+ - 1) << 1$, $\tilde{r} = r/r_0$. Using eq(4) and the inversion constraint $\mathrm{d}t_- = -\mathrm{d}t_+ = -r\mathrm{d}\bar{\eta}$, eq(3) gets transformed to

$$-\,\mathrm{d}x_1 = \tilde{r}\lambda(1 + \tilde{r}\nu T_1)\,x_1 \mathrm{d}T_1 \tag{5}$$

where $T_1 = \ln \eta_1$, $\eta_1 = 1 + \bar{\eta}_1 = 1 + \nu(\eta_+ - 1)$ and $\lambda = \kappa/\nu$, when we make use of $\ln(1 + \tilde{r}\bar{\eta}_1) \approx \tilde{r}\bar{\eta}_1 \approx \tilde{r}T_1$. This equation, valid close to $\bar{\eta} = 0, (\eta = 1)$, and self similar to eq(2) , describes the small scale evolution of the first generation fluctuation $x_1$. (One can, indeed, recast eq(5) exactly to the form eq(2) in the stochastic time variable $t_1 = \tilde{r}T_1$.) The (-) sign is a signature of inversion. Note that this also avoids the possibility of a backward flow of time at the expense of deflecting the direction of system evolution. As time $\eta$ continues to flow from $\eta_+ \approx 1$ onwards, dragging $\bar{\eta}_1$ along with, the small scale fluctuation $x_1$ also gets amplified following eq(5) and assumes the status of the original system variable $x$ over time $\eta \sim O(e(1 + \nu) - \nu)$, when a second generation transition to the scale $T_2$ becomes permissible. Note that the self similarity of eq(5), relative to the time variable $t_1$, with eq(1) tells that $T_1$ would act as the ordinary time for $T_1 \in (0, 1)$ for the evolution of $x_1$. Factoring $x_1 = e^{-\lambda t_1}x_2$, one thus gets the 2nd generation replica of eq(1) for $x_2$ in the time variable $T_2$ and hence this method of self replication over scales $T_n$ could continue ad infinitum. An infinite string of iterations, as above, thus leads to a new solution of eq(1) in the form

$$x \propto e^{\eta - \tilde{r}\lambda(T_1 - T_2 + \ldots)} \tag{6}$$

Note that $T_2 = \ln \eta_2$, $\eta_2 = 1+\nu(T_1-1) = 1+\nu(\ln(1+\nu(\eta-1))-1))$ and hence the nth generation scale $T_n$ is related to $\eta$ by $n$ nested natural logarithms. Consequently, the over all fluctuations in the system $x_f \propto x/x_0 = (\frac{\eta_2\eta_4\ldots}{\eta_1\eta_3\ldots})^{\tilde{r}\lambda}$, incorporating influences of all scales, has the asymptotic form $x_f \sim (\frac{\ln\eta\ln\ln\ln\ln\eta\ldots}{\eta\ln\ln\eta\ldots})^{\tilde{r}\lambda} \equiv \eta^{-\tilde{r}\mu}$, as $\eta \to \infty$. Here, the exponent $\mu = \lambda(1 - \frac{\ln\sigma}{\ln\eta})$, $\sigma = \frac{\ln\eta\ln\ln\ln\eta\ldots}{\ln\ln\eta\ldots}$, is a slowly varying function of $\eta$. A few remarks are in order.

1. The choice of the moment $t = 1$, around which the evolution is probed, is for the sake of convenience. For any point $t_0 \in (0, 1)$, the analysis proceeds with the rescaling $t \to t/t_0$, $\kappa \to t_0^2\kappa$ in eq(3).

2. Because of the nested logarithms, contributions from higher order scales are felt slower and slower. As stated above the first sign of fluctuation is surfaced only if the system is allowed to evolve over a period $\eta_f \sim 1 + 10^{-m}r_0^{-1}$. As an example, in a 'universe' where only the first order accuracy ($m = 1$) is allowed, the fluctuation is first noticed around $\eta_f \sim 2$ for a $r_0 \sim 0.1$. More generally, for a $r_0 = 10^{-(m+s)}$, $\eta_f$ could be arbitrarily large, for a large value of $s$, even in the limit $m \to \infty$ of infinite accuracy.

3. It is natural to interprete the new solution eq(6), with self similar fluctuations over the mean solution $x_0$, as a (random) fractal solution of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \tag{7}$$

under time inversions. This is, however, in contrast to our aim which was to find such a solution for eq(1), so that stochastic fluctuations would have been around the standard solution $x_s$. In the following, we resolve this dichotomy in a more general framework of fractal time, where the parameter $\lambda$ is identified with an 'apriori' scale factor of time, rather than a system variable. We also show how the correct fluctuation pattern is obtained when $\lambda$ acts as a true system variable.

# 4 Fractal time

We have shown how a stochastic behaviour could be injected to time via inversion, while time inversion is interpreted as a consequence of the practical limitation of exact measurability of a duration. ( Stated more precisely, the possibility of time inversion raises the practical measurement limitation to the level of a theoretical principle.) The stochastic nature of time already endows time with fractal -like characteristics. To investigate fractal properties in more details, let us begin by writing an *ansatz* for the 'physical' fractal time $t$ as an implicit random function of the ordinary time $\eta : t_{\pm}(\eta) = \eta(1 \pm \kappa\eta t_{\mp}(\eta^{-1}))$. Here, $\kappa > 0$ stands for an arbitrary, small but random, parameter. The nontrivial second factor would be responsible in determining the small scale structure of the physical time in the neighbourhood of every point $\eta$. The choice of sign $\mp$ in $t$ in the factor indicates explicitly the possibility of inversion near $\eta = 1$ (see below). Note also that $t_{\pm}$ mimics the notation of Sec.2-3, thereby splitting every point of $\eta$ axis into an equivalence class $\{t_{\pm}\}$ of infinite number of finely separated points. Since each member of the class is a function of $\eta$, it has, at least, the cardinality of continuum (c.f., remark 3, Sec.2). We show below that the actual cardinality is $2^c$.

As it turns out, the ansatz represents *a new class of exact, stochastic solutions* to the equation $\frac{dx}{d\eta} = 1$. To verify this, we note (suppressing the distinction temporarily) that, by symmetry, both $t/\eta$ and $\eta\tilde{t}$, $\tilde{t} = t(\eta^{-1})$ satisfy coupled equations of the form $\alpha = 1 + \kappa\beta$ and $\beta = 1 + \kappa\alpha$, hence $t(\eta)/\eta = \eta\tilde{t}(\eta)$ for all $t$. Noting that $\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\eta} = -\eta^{-2}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\eta^{-1}}$, we get $\frac{\mathrm{d}t}{\mathrm{d}\eta} = (1 + \kappa\eta\tilde{t}\ ) + \eta\kappa\tilde{t} - \kappa\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\eta^{-1}}$, so that ,$(\frac{\mathrm{d}t}{\mathrm{d}\eta} - \frac{t}{\eta}) + \kappa(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\eta^{-1}} - \eta\tilde{t}\ ) = 0$. It thus follows, $\kappa > 0$ being arbitrary, that

$$\eta \frac{\mathrm{d}t}{\mathrm{d}\eta} = t \tag{8}$$

which is nothing but the desired equation in logarithmic variables. Note that for a nonrandom real parameter $\kappa$, one retrieves the standard solution $t = (1-\kappa)^{-1}\eta$. For a random $\kappa$, which arises naturally in the context of the present 'local' definition of inversion, we, however, get a new *random (fractal) solution*, which matches (approximately) with the standard linear solution only in the mean. We emphasise that the new solution is an *exact* solution of eq(8). Because of its inherent scale-free nature, the solution must possess nontrivial fractal characteristics. Indeed, eq(8) tells that $\ln t$, and hence $t/\eta$, must be a function of $\ln \eta$, so that $t = \eta\phi(\ln \eta)$. Here, $\phi(\ln \eta) = c(1 + \kappa\phi(\ln \eta^{-1}))$, and represents a nontrivial (random fractal ) solution of $\frac{\mathrm{d}x}{\mathrm{d}\eta} = 0$, $c$ being a real constant.

Let us note that a straightforward iteration of the ansatz in the form $t/\eta = 1 + \kappa + \kappa^2 + \ldots$ would be, in general, misleading because this geometric series in $\kappa$ apparently hides the slow time dependence that is always present in any finite approximants of this infinite series. We have already shown how such a slow, residual time dependence in the $n$th approximant $S_n = 1 + \kappa + \kappa^2 + \ldots + \kappa^n \eta \tilde{t}$ can influence the dynamics over time $\eta \sim < \kappa >^{-n}$, because of local time inversions.

*To explore the role of golden mean* $\nu$, *ab-initio* in the present context, let us now reintroduce signs '$\pm$' to distinguish the variables $t_{\pm}$. Let $t_+/\eta = 1 + r\eta_1 t_-(\eta_1^{-1})$ and $t_-/\eta = 1 - r\eta_1 t_+(\eta_1^{-1})$, where $\eta_1 = k\eta$, $\kappa = rk, k$ is an ordinary constant, and $r\,(\sim 1)$ is a random variable, analogous to one in Sec.3. Note that the present form is slightly general from above, but, nevertheless, solves eq(8), because of its scale free nature. This scale free property now tells that the limit of $t_+(\eta)/\eta \equiv \phi(\eta)$ as $\eta \to \eta_0, 0 < \eta_0 < \infty$ is independent of $\eta_0$. As a consequence, $\phi(\eta)$ is a *universal* random function, defined in the vicinity of $\eta = 1$, so that $t_+ = \eta\phi(\eta)$, $\phi(\eta) = 1 + r\phi(\eta_1^{-1})$. Note that for a sufficiently small $k$, the variation of $\phi$, which always remains of the order O(1), is very small. Note that $t_{+1}(\eta) = kt_+(\eta) = t_+(\eta_1)$, by definition.

Let us now recall that $\eta_1 = 1$ is an 'ideal' point in the ordinary time axis, in contrast to the 'physical point' $t = 1$, an equivalence class of members of the form $\{t_{\pm}\}$ in the geometrical axis of the physical time. Now, as $\eta$ approaches $\infty$ crossing $k^{-1}$, the rescaled variable $\eta_1$ crosses $\eta_1 = 1$, running over points such as $\eta_{1-} = 1 - \sigma$ to $\eta_{1+} = 1 + \sigma$. An ordinary time inversion $\eta_{1-} = \eta_{1+}^{-1}$ (c.f., remark 3,Sec.2), now induces a random inversion in the physical time $t_-(\eta_{1-}^{-1}) \to t_-(\eta_{1+}) : rt_-(\eta_{1+}) = \eta_{1+}^2/t_+(\eta_{1+})$, so that $r\phi(\eta_1^{-1}) = 1/\phi(\eta_1)$. Note that the exact moment of inversion is uncertain because of the inherent randomness in the physical time due to the r.v. $r$. Consequently, one obtains $t_+/\eta = 1 + \eta_1/t_+(\eta_1)$, with $\eta_1 = 1 + \sigma, \sigma \to 0$. It thus

follows that $t_+ = \phi\eta$, $\phi(\eta) = 1 + \nu$, which is true not only for $\eta \to \infty$, but, in fact, for *any* $\eta$ because of the universality of $\phi$.

*To continue further, let us note that the above result, although proves the unique role of the golden mean number in the framework of fractal time, also presents us with a riddle.* Apparently, one would like to conclude that this solution reproduces the standard solution $t = c\eta$ of eq(8). However, the emergence of this unique special value $\nu$ is unclear in the ordinary framework. Recall that the variables $t_\pm$ are intrinsically random. Further, no where in the above analysis we have taken mean values to eliminate the underlying randomness. The only reasonable conclusion would therefore be the following: *the golden mean number $\phi$ represents a universal random fractal function which is responsible for small scale random fluctuations in the physical time $t$.*[3]

To explain the assertion in detail, we need to proceed in a number of steps, establishing a number of key results.

Let us begin by noting that the ordinary solution of eq(8) defines a 1-1 ( the identity) mapping $R \to R$ of the ordinary real set. The new class of random scale free solutions now defines an extension of the ordinary real set to the 'physical ' real set $P$, say. The extended solution space of linear ordinary differential equations now consists of solutions which are (i) infinitely differentiable and (ii) those which are finitely differentiable, stochastic functions. We shall verify shortly that the new implicitly defined functions are first order differentiable, with discontinuous second derivatives, at the moments when $\eta$ changes by inversions. Because of this extension, it is natural to expect that the physical set $P$ contains new members not available to $R$. To show that $P$ indeed has nontrivial numbers, let us first distinguish a physical number $K_p$ from an ordinary positive real number $K$, where $K_p = K + N$, $N$ being the neighbourhood of 1 consisting of random physical numbers (fluctuations). Thus, a physical number $K_p$, representing a nontrivial equivalence class, is a non-singleton subset of $R_+$, the set of nonzero positive reals. Further, by definition $K_p = K\phi(\eta)$.

Let $P(\tilde{R})$ denote the power set of $\tilde{R} = R_+ \cup (-1, 0)$, $\tilde{P}$, being the corresponding extension, and $N$ the set of discrete subsets in $\tilde{R}$. Then the cardinality of $N$ equals, $c$, the cardinality of continuum. Let $g : N \to (-1, 0)$ denote a natural 1-1 correspondence. Now, there exists an injection $f_1 : P(\tilde{R}) \to \tilde{P}$, defined by $f_1(A) = g(A)$, when $A \in N$; $f_1(A) = K$, when eq(8) admits infinitely differentiable solutions in $A$; and $f_1(A) = K_p$, otherwise. Conversely, one also finds an injection $f_2 : \tilde{P} \to P(\tilde{R})$, defined by $f_2(K) = \{K\}$, but $f_2(K_p) = \{K, B\}$, where

[3] From now on, we denote by $\phi$ this stochastic time-like feature of the golden mean. By the symbol $\nu$ we, however, continue to refer to the usual 'non-random' number $\frac{\sqrt{5}-1}{2}$.

$B = (0, 1)$ is the interval where $K_p$ is defined. Hence, by the Schroeder-Bernstien theorem [7], the cardinality of the physical set $P$ equals $2^c$. Note that the mapping $g$ renders all possible discrete subsets of $P$ unphysical, transferring them to (-1,0). Further, eq(8) admits infinitely differentiable solutions, over a given, *apriori* scale denoted as $\eta$, provided the possibility of random scale dependence via inversions is neglected. Consequently, the set $R$ is realized in $P$ in an approximate sense. Moreover, there exist an uncountably more nontrivial elements in the physical set in comparison to the real set. This means, in particular, that there exist physical numbers ( in the form of fluctuations) which are infinitely smaller than any nonzero ordinary real numbers. By inversion, one then concludes that there exist infinitely large physical numbers greater than any real number. Clearly, all the set theoretic results, valid in $R$ , get carried over to the physical set. In particular, the physical set is partially ordered, when $k_p \leq K_p$ means $k \leq K$.

Now to construct a nontrivial infinitely small fluctuation, let us recall that a physical number is associated with an accuracy limit. Let $k_p = rk(\eta^{-1})\eta$ be determined with an accuracy $10^{-m}$. Let $\sup[1/k] \approx 10^{(m+s)}$. Here, [.] denotes the greatest integer function, and the supremum is defined on a bounded interval of $\eta$. Then, as $m \to \infty$, $s$ large, but finite, $k_p = 0$, in the ordinary sense, for finite $\eta$. But in the sense of a 'physical limit' $\eta_1 = 10^{-m}\eta \to 1$, $m \to \infty$, $\eta \to \infty$, one gets an arbitrarily small random number, by allowing $s$ to assume larger and larger values, but never exactly allowing $s = \infty$. Because of randomness, $k_p$ is not an ordinary real number. The physical limit tells that the ideal condition of infinite accuracy is realised only in an infinitely distant time. In other words, any measurement process over a finite period of time can achieve only a finite degree of accuracy. We note that the physical set $P$ has a structure analogous to the nonstandard real number set [5]. A more detail investigation of the relationship between the two approaches will be considered separately.

Next *we show how such an infinitely small fluctuation $k_p$ can have nontrivial influence at the level of the ordinary scale, thereby re-deriving, in an alternative way, the small scale time evolution in the golden mean $\nu$.* Let us rewrite the physical time $t$ in the form $t = \eta\phi(\eta)$, $\phi(\eta) = 1 + k_p\phi(\eta_1^{-1})$, $\eta_1 = k\eta$. Since $t$ must be an exact solution of eq(8), one obtains $k_p \frac{\mathrm{d}\phi}{\mathrm{d}\ln\eta_1} = 0$. It thus follows that in the ordinary scale of $\eta : \frac{\mathrm{d}\phi}{\mathrm{d}\ln\eta} = 0$, and hence $\phi$ is an ordinary constant (having no evolution). However, this conclusion may not necessarily be true at the level of the smaller scale $\eta_1$. In fact, as $\eta_1$ grows to order O(1), $\frac{\mathrm{d}\phi}{\mathrm{d}\ln\eta_1}$, need not vanish, since $k_p$, though nonzero, becomes vanishingly small provided $\frac{\mathrm{d}\phi}{\mathrm{d}\ln\eta_1} \sim$ O(1). To verify this, note that $\phi(\eta) - 1 = \bar{\eta}_1(\phi(\eta_1^{-1}))$ ($\phi < 1$ as long as $\eta_1 < 1$), $\bar{\eta}_1 \approx 0$, so that in the limit of $\eta_1 = 1 + \bar{\eta}_1 \to 1$,

$$\frac{\mathrm{d}\phi(\eta_1^{-1})}{\mathrm{d}\ln\eta_1} = \phi(\eta_1^{-1}) \tag{9}$$

which is of order O(1), and one retrieves a self similar replica of eq(8), over the scale $\eta_1$. Note that, as $\eta_1 \to 1^-$, $\phi$ behaves as a small scale linear time so that $\phi - 1 \approx -\mathrm{d}\phi$. But as $\eta_{1-} \to \eta_{1+}$, by inversion, (-) sign cancels so as to reproduce eq(9). It is now easy to verify the continuity of the first derivative at $\eta_1 = 1$, since $-\frac{\mathrm{d}\phi(\eta_1^{-1})}{\mathrm{d}\ln\eta_1^{-1}} = \frac{\mathrm{d}\phi(\eta_1)}{\mathrm{d}\ln\eta_1} = \phi$. *The continuity of the second derivative can not, however, be maintained because of the sign difference injected by inversion.*

Note that the above analysis leads us to the fact, by yet another route, that the golden mean $\nu = \phi - 1$, must have small scale *intrinsic* time evolution. To state more clearly how a time sense gets attached to $\nu$, let us note that the intrinsic time must flow, in the neighbourhood of $\eta(= 1)$, in the form $\phi(\eta) = 1 + k_p\phi(\eta_1^{-1})$, till a random inversion near $\eta = k^{-1}$ carries it to the smaller scale $\eta_1$, to assume the form $\phi(\eta) = 1 + k(\phi(\eta_1))^{-1}$. As a consequence, a (linear) time sense, self similar to the scale $\eta$ is generated over the scale $\eta_1$ , which persists upto order $\eta_1 \sim k^{-1}$, preparing it for yet another replication on the second generation scale $\eta_2$ and so on, leading to an infinite continued fraction $\phi_k = 1+[k; k; k; \ldots]$ representation of the intrinsic time flow, as the zeroth generation ordinary time $\eta \to \infty$, exploring longer and longer scales. Letting $k = 1$, one gets *the golden mean flow of time*. Note that the intrinsic sense of time is derived from the infinitely slow random unfolding of cascaded scales hidden in the form of an infinite continued fraction. Among all these time-like continued fractions $\phi_k$, the golden mean continued fraction is distinguished by its slowest possible convergence (unfolding) rate. We now show that $\phi_k - 1 \to \phi$ as $\eta \to \infty$. Note that after $n$th inversion, $\eta$ reaches the scale $\eta_n$. Let $n_\infty$ denote the smallest infinitely large natural number, exceeding all real numbers, in the physical set $P$, so that $\eta_\infty = k^{n_\infty}\eta$. One exhausts all possible ordinary scales by utilising all the available $n_\infty$ number of inversions so that the final replication leads to $\phi(\eta_\infty) = 1 + \frac{1}{\phi(\eta_\infty)}$ , yielding $\nu$. It thus follows that although, in general, time may flow following the steps of $\phi_k$, this scale dependent flow can in fact continue at most upto a finitely many scales (in the context of physical time). *The intrinsic flow, exploring smaller and smaller scales at slower and slower rates, as the ordinary zeroth generation scale $\eta \to \infty$, would finally converge to the slowest possible golden mean flow.* Note that once *the golden mean flow* is reached, no further scale replication is allowed, since all the higher order scales would be physically indistinguishable ($\eta_{\infty+1} = \eta_\infty$), from the stand point of the ordinary scale $\eta \sim 1$. However, because of self similarity, any scale $\eta_n$ could be considered as the zeroth generation scale, the process of exploring the golden mean flow of time would remain unaltered.

We close this section with a few more remarks, highlighting a number of important features

of the golden mean.

1. The golden mean time sense is intrinsic, since it is independent of an *apriori* time. Note that $\phi(\phi(.)) = \phi(.)$, where $(.)$ indicates that $\phi$ could be a function of any $t_n$. In fact, the equation means $\phi = \phi(\phi)$, thus eliminating the Newtonian external time $\eta$. (Recall that to avoid any confusion, we choose $\phi$ to indicate the slow scale dependent (logarithmic) variation in the golden mean, while $\nu^{-1}$ denotes the usual constant value of it, over a well defined given scale.) However, an approximate, 'coarse-grained' Newtonian sense of time is realised over a scale, when the small scale fluctuations, because of local inversions, are ignored. In the following, we denote by $\phi(\phi)$ the set (equivalence class) of all possible intrinsic variables of the form $\{\phi(t_n)\}$.

2. So far, we have not spelled out the form of the random variable $r$. A probability distribution satisfying the constraint $< r^n >\approx r_0^n$, $< r >= r_0$ can be written as follows. Let the sample space of $r$ be $\{r_0^{s+1}\}$, $s = 0, 1, 2, \ldots$. The corresponding discrete probability function is defined by $p(r) = e^{-r_0^m} \frac{r_0^{ms}}{s!}$, $m$ being a positive integer. The distribution is Poisson-like, but not exactly the same, because of the special sample space, generating *the infinite set of scales*. Let $r_0 \lesssim 1$, so that $r_0^m << 1$, for a suitably large $m$. Thus $< r^n >= r_0^n p(r_0), p(r_0) = e^{r_0^m (r_0^n - 1)} \approx 1$. Further, any number $r_0$ can be written as $r_0 = \nu^a$, $a = \log_\nu r_0$. Thus all the free parameters in the definition of the fractal time $t$ is determined self consistently in terms of $\nu$. The parameter $k = 1$, since the scale factor is included in the sample space. One thus fixes $\lambda$ in eq(1) and (6) as $\lambda = r_0$.

3. The golden mean $\nu$ remains constant over a scale $\eta$, but, nevertheless, enjoys intrinsic randomness. Because of the equality $\nu = (1 - \nu)\Sigma\nu^{2n}$, $\nu$ is realized as the expectation value of a r.v. $r$ with sample space $\{\nu^n\}$ of scales, the corresponding probabilities being $\{p(r)\} = \{(1 - \nu)\nu^n\}$. Note that $\nu$ here denotes the exact value represented as $\frac{\sqrt{5}-1}{2}$. Let us consider the sample space $\{S_{m+n}\}$ of yet another r.v. $r_\nu$, for a sufficiently large $m$, where $S_n$ is the $n$th approximants of the above series. The corresponding probability distribution is $p(r_\nu)$. Consequently, $r_\nu$ realizes higher precision values of $\nu$ with lower and lower probabilities. Clearly, $\nu \equiv r_\nu$. This proof, being purely of number theoretic origin, tells that measurements in a physical universe are inherently uncertain, since any measurable quantity is a multiple of $\nu$. Note that the evaluation of the exact $\nu$ needs a measurement process defined over an infinite amount of time, thus making it physically impossible. The choice of the probability distributions may, however, appear arbitrary. In Sec.6.2, we show how these discrete distributions relate naturally to a universal density function for the class of fluctuations considered in this work.

# 5 Applications

In the following applications of the fractal time, we show how the formalism yields (i) a natural resolution to the generic $1/f$ signal problem [1], and (ii) the universal probability density observed in the Bramwell- Holdsworth- Pinton (BHP) fluctuations [2].

To begin with let us first reconsider eq(1) in the light of fractal time $t$. Recalling $t = \eta(1 + r\phi(\eta^{-1}))$, $\mathrm{d}t = (1 + r\phi)\mathrm{d}\eta$, since $\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = 0$. The equation (7) in the 'physical' time $t$ is re-expressed in the Newtonian time as

$$\frac{\mathrm{d}x}{\mathrm{d}\eta} = (1 + r\phi(\eta^{-1}))x \tag{10}$$

An exact solution of this equation, eq(6), as an implicit function of $\eta$ has already been obtained. An analogous form of the solution is now written as $x = e^{(\eta+\alpha(\eta))}$ $\alpha(\eta) = r\int \eta \mathrm{d}\phi(\eta^{-1})$, when eq(9) is used. Note that the infinite series of scales $T_n$, with alternating signs, provides an explicit representation of the intrinsic time variable $\phi$, when the system is allowed to evolve over all the available scales in an infinite period of the ordinary time $\eta$. The exponent $\alpha$ thus leads to the exponent $\mu$ of eq(6), when the integration is performed successively over scales between two consecutive moments of inversion. It thus follows that the method of Sec.3 actually yields the fractal solutions to eq (7), when $\kappa$ is identified with one of the scales $\nu^n$. Moreover, a solution of eq(1) when $\kappa$ is treated as a system variable can be obtained as $x = e^{\eta+\frac{1}{2}\kappa\eta^2-\beta(\eta)}$, where $\beta(\eta) = r\int(\eta+2k\eta^2)\mathrm{d}\phi(\eta^{-1}))$, neglecting $\mathrm{O}(r^2)$ corrections. Note that both the solutions indicate almost identical late time power law fluctuations.

## 5.1 1/f spectrum

To calculate the spectrum of the stochastic fluctuations, present universally in linear equations of the form eq(1), we need to estimate the late time asymptotic form of the corresponding two-point correlation function $C(\eta) =< x(0)x_f(\eta) >=< x_f(\eta) >$, since $x(0) = 1$. Assuming that $|\lambda| \approx \nu^n$ (say), it follows that $C(\eta) \sim \eta^{-\mu}$, and hence the spectrum has the form $S(f) \sim 1/f^{1\mp\mu}$, $\mu$ being the slowly varying function of Sec.3. Clearly this should be the generic form of the spectrum for a general dynamical system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h(t)x \tag{11}$$

where the time dependence in $h$ may have nonlinear influences: $h(t) = h(t, x)$. However, the nature of explicit nonlinearity in a system is expected to get reflected in the exponent $\mu$. The generic logarithmic correction in $\mu$ provides significant insights into the late time features of

the dynamics, which might get revealed in a time series over a number of different scales. One example is treated below.

## 5.2 Universal probability function

Recently, a universal pattern of self similar fluctuations have been reported to occur in many Natural processes. Subsequently, a generic probability density function (PDF) of the form

$$P(t) = Ke^{at-ae^t} \tag{12}$$

is shown to be respected by the underlying dynamics [2], of apparently unrelated systems. Here, $t$ is a relevant fluctuating variable. Let us note that the solution eq(6) denotes the universal fluctuation pattern, at least, for those Natural processes which satisfy an equation of the form eq(11). We now show how the above PDF is naturally realized for this universal fluctuation.

Let us begin by noting that the infinite alternating series of scale dependent terms in eq(6) gives the complete fluctuation spectrum $x_f$ of a linear system, which is self similar over all these scales and is revealed over an infinite period of the ordinary time $\eta$. However, the first two terms in the series are sufficient to capture the *generic* features of the fluctuation $x_f$ , because the scale generating r.v. $r$ (c.f., remark 2, Sec.4) induces a higher order (stochastic) scale dependence on each of the scales $T_n$, thereby inscribing a complete replica, of the total fluctuation $x_f$ over the scale $T_1$ (say), even in a finite period of $\eta$. Further, the (-) sign between two consecutive scales $T_1$ and $T_2$ is a nontrivial signature of inversion. Thus it suffices for us to consider a renormalised fluctuation of the form $\tilde{x}_f \propto e^{-r(T_1-T_2)}$ $(\lambda = r_0)$, for a finite $\eta$. Accordingly, the generic PDF corresponding to $\tilde{x}_f$ should be identical with the same for $x_f$.

Let, for definiteness, that the random scales $r = \{\nu^{n-1}\}$ be distributed with probabilities $p(r)$ introduced in remark 3, Sec.4. Because of the logarithmic scale dependence $T_2 = \ln(1 + \nu(T_1-1))$, one gets $\tilde{x}_f \propto e^{r(T_2-(1+\nu)e^{T_2})}$. Dropping the higher order scale dependence introduced by the $\nu$ dependent factor, we get finally $\tilde{x}_f \propto e^{r(T_2-e^{T_2})}$. Now in a practical situation, the fluctuation $\tilde{x}_f$ is represented in the form of a time series record over a period of time. The probability that a randomly sampled observation $\tilde{x}_{fn}$ is drawn from the scale $r_n$, in the sample space of $r$, now equals the product of the probability of selecting the scale $r_n$ and the conditional probability that the observation actually comes from the said scale, given that the scale has been chosen already. But the conditional probability is nothing but the correlation function $C(T_2) =< \tilde{x}_{fn}(T_2)\tilde{x}_f(0) >=< \tilde{x}_{fn}(T_2) >$, $\tilde{x}_f(0) = 1$, $\tilde{x}_{fn} \propto e^{r_n(T_2-e^{T_2})}$. Thus the probability of drawing a random sample from the scale $r_n$ equals $P(r_n) \propto p(r_n)e^{r_n(T_2-e^{T_2})}$. Hence the grand universal probability that the time series reveals the whole spectrum of fluctuations over all

possible scales is obtained as

$$P_u \propto \Sigma_0^\infty P(r_n) = (1-\nu)\Sigma_0^\infty \nu^n e^{(1+\nu)\nu^n (T_2 - e^{T_2})} \tag{13}$$

Clearly, apart from a multiplicative factor, which could be fixed from the normalisation condition, the zeroth order term of this infinite series representation of the universal PDF agrees well with that in ref.[2], which was obtained from an approximate argument (the factor $a = \pi/2$ in the exponential gets replaced here by $1+\nu$). The higher order terms in the infinite series represent corrections, which are likely to give more accurate fits of the time series records for Natural processes, as noted in ref.[2].

Now to explain the reason of the matching, we note that the factor $1+\nu$ in eq(13) actually realises $T_2$ as $T_1 (\equiv (1+\nu)T_2)$ and $T_1$ as $\eta$. In the case of an explicitly nonlinear system given by eq(11) with nonlinearity coupling $k \sim 1$, the system experiences fluctuations at a time $\eta \sim 1$. The scale $T_1$ then corresponds to the 2nd generation fluctuation in a corresponding time series record. It thus follows that the zeroth order PDF would have the form $e^{T_1 - e^{T_1}}$. Let us note that the moments of the model fluctuating variate in ref.[2] possesses the generic property $< r^n > \propto r_0^n$. However, the corresponding PDF is obtained from a quantum field theoretic consideration of a critical magnetic model which is based purely in the Newtonian time frame. Now, the relationship between an 'ordinary' dynamical variable $Q_0$, following an evolutionary equation of the form eq(11), but in the ordinary time, and the corresponding 'physical' variable $Q_p$ is given by $Q_0 \propto Q_p^\nu$, since in the logarithmic scale $\ln t = (1+\nu)\ln \eta$. An application of this conversion rule thus leads to

$$P_{BHP} \propto e^{(1+\nu)(T_1 - e^{T_1})} \tag{14}$$

which completes the derivation of the BHP probability function.

To proceed further, let us now re-derive the generic PDF from an alternative method. This will reveal a subtle relationship between the universal PDF and the set of discrete distributions we have chosen as examples. Let $\eta_n = \nu^n \eta$, where $\eta \in (0,\infty)$ be nonrandom, and $\nu$ denote an approximate value of the golden mean. The scales $\eta_n$ are random, because of the randomness in $\nu$ and is assumed to follow the distribution of remark 2, Sec.4 ($m=1$), so that $\eta_1$ is realized with probability $P_1 = e^{-\nu}\nu$. Since $\nu$ is approximate, it will now undergo evolution in the physical set following, for instance, the scale free representation $\nu_p = \nu\phi(\eta_1^{-1}) = \nu/(1+\phi(\eta_{11}^{-1})) = \nu/(1+/\{1+\phi(\eta_{12}^{-1})\})$, etc. We may assume that the linear sense of time generated by $\phi$ at the scale of $\eta$ is denoted by $\eta$ itself. As $\eta$ flows from $\eta \approx 0$ slowly, the intrinsic evolution in $\nu$ splits the scale $\eta_1$ into an infinite set of tiny scales $\eta_{1n} = \nu^n \eta_1$, each of which will further

undergo finer levels of subdivisions, and so on. The stochastic evolution of $\nu$ thus fractures the scale $\eta_1$ at the neighbourhood of every point, thereby raising it to the level of a continuous r.v. with the PDF $P_1(\eta_1) \propto e^{-\nu\phi(\eta_1^{-1})}\phi(\eta_1^{-1})$, a Gamma distribution. We note that El Naschie [8] has already used a Gamma distribution to derive the Hausdorff dimension of the fluctuating (Cantorian) spacetime as $4 + \nu^3$. He has also indicated how the mass spectrum of all known elementary particles could be determined using the golden mean $\nu$. (The uncertainty exponent of the fractal time is obtained independently by the author as $\nu$ [10].)

All the higher order scales will similarly undergo infinitesimal fluctuations and hence finally be distributed following the above Gamma density function, since $\nu_p^n = \nu^n \phi(\eta_n^{-1})$. Note that all these $\phi(\eta_n^{-1})$ functions correspond to different scale dependent realizations of the same universal function $\phi(\phi)$ in the limit of infinite time. Letting $\nu\phi = e^{\tilde{\phi}}$, we reproduce the universal PDF for scale free (self similar) fluctuations, $P(\tilde{\phi}) \propto e^{\tilde{\phi} - e^{\tilde{\phi}}}$. But $\tilde{\phi}$ is again a realisation of $\phi(\phi)$, by eq(9) and remark 1, Sec. 4.

# 6 Concluding remarks

The extension of the real set to the physical set provides a dynamical representation of the number system, each member of which is associated with an equivalence class of fluctuating elements separated by infinitesimally small scales. The fundamentally stochastic nature of the golden mean number renders accurate measurements in the physical set impossible. Consequently, the physical universe based on the physical set would consists of intrinsic changes and fluctuations. It is hard to imagine any fundamentally constant physical quantity in this universe. The real number set is an incomplete realization of the physical set, when the possibility of infinitesimal changes by local inversions are ignored. However, because of the new exact class of solutions, $P \equiv R$. In the midst of all these changes and approximations, there exists, however, one symbol of perfection in the form of the golden mean equation $\phi(\phi)^2 + \phi(\phi) = 1$, being engraved fundamentally in the formalism of *the SL(2,R) Calculus*. A more detailed, in depth analysis of this Calculus, will be presented in a subsequent paper, where the status of well known theorems such as the Picard's existence and uniqueness theorem will be examined. Let us only remark here that the present class of solutions are not in contradiction with the Picard's theorem, the scope of which gets extended in the SL(2,R) formalism. The present formalism is likely to initiate a new approach in understanding the origin and dynamics of the nonlinear phenomena in Nature.

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